Sec 5.5 Nonhomogeneous Eqn (part 1)

Recall: ① $y'' + p(x)y' + q(x)y = f(x)$ non-homog. if $f \not\equiv 0$ (f is not constant 0)

associated homog eqn $y'' + p(x)y' + q(x)y = 0$

General solution for ① is

$$y = \underbrace{\boxed{c_1 y_1 + c_2 y_2}}_{\text{"}y_c\text{"}} + \boxed{y_p}$$

solution to assoc. homog eqn ("complementary function")

"particular solution (for ①) to account for $f \neq 0$"

so gen solution for nonhomog. linear ODE is

$$\boxed{y = y_c + y_p}$$

(similar for 3rd, 4th, ... order)

Example for method:

Find general sol for $y'' - 4y = 3x$ ②

$$y = y_c + y_p$$

Step 1 Compute y_c $\quad (y'' - 4y = 3x^2)$

assoc. homog eq. is $y'' - 4y = 0$

ch. eq is $r^2 - 4 = 0$

$\Rightarrow r = \pm 2$

Gen sol for. assoc. homog eqn is

$$\underline{y_c = c_1 e^{-2x} + c_2 e^{2x}}$$

Step 2 Compute/find y_p, using a "trial function"

(this is the "method of undetermined coefficients")

Idea:

- need y_p to satisfy $\underline{y_p'' - 4y_p = 3x^2}$
- want to have $y_p, y_p', y_p'', \ldots$ all add/subtract/mix in $(y_p'' - 4y_p)$ to result in $\underline{3x^2}$
- If we "trial"/"guess" $y_p = e^{kx}$ (k TBD), then

$y_p = e^{kx}$, $y_p' = ke^{kx}$, $y_p'' = k^2 e^{kx}$; all these contain e^{kx}. Thus, adding/mixing $y_p'' - 4y_p$ will probably still have e^{kx}, and is unlikely to result in $3x^2$

- It is better/best to pick something for y_p to have same "form" as $f(x) = 3x^2$.

$f(x) = 3x^2$ (degree 2 polynomial $3x^2 + 0x + 0$)

so let's trial $y_p = \underbrace{A} + \underbrace{B}x + \underbrace{C}x^2$ (2nd-deg. poly)

"undetermined coeffs"

$$y_p = A + Bx + Cx^2$$
$$y_p' = B + 2Cx$$
$$y_p'' = 2C$$

$$\Rightarrow \quad y_p'' - 4y_p = 3x^2$$

$$2C - 4(A + Bx + Cx^2) = 3x^2 + 0x + 0$$

$$-4Cx^2 - 4Bx + (2C - 4A) = 3x^2 + 0x + 0$$

$$\Rightarrow \left.\begin{aligned} -4C &= 3 \\ -4B &= 0 \\ 2C - 4A &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} &C = -3/4 \\ &B = 0 \\ &4A = -6/4 \Rightarrow A = -3/8 \end{aligned}$$

thus, $\boxed{y_p = -\frac{3}{8} - \frac{3}{4}x^2}$ is a solution for $y'' - 4y = 3x^2$

Step 3 : Full gen. sol for $\rightarrow$ is

$$y = y_c + y_p = \boxed{c_1 e^{-2x} + c_2 e^{2x} + \left(-\frac{3}{8}\right) - \frac{3}{4}x^2}$$

Note: Why include y_c?

with $y_p = -\frac{3}{8} - \frac{3}{4}x^2$, we know

$$y_p'' - 4y_p = 3x^2$$

- y_c, by defin, satisfies $y_c'' - 4y_c = 0$
- So then $y_c + y_p$ as solution? $y'' - 4y = 3x^2$

$$(y_c + y_p)'' - 4(y_c + y_p) = 3x^2$$

$$y_c'' + y_p'' - 4y_c - 4y_p = 3x^2$$

$$\underbrace{(y_c'' - 4y_c)}_{0} + \underbrace{(y_p'' - 4y_p)}_{3x^2} = 3x^2$$

- Need to include y_c to make complete/all solution(s)

Ex: $y'' - y' - 2y = 2e^{3x}$

step 1: compute y_c, solution for $y'' - y' - 2y = 0$

$$\cdots \quad y_c = c_1 e^{-x} + c_2 e^{2x}$$

Step 2: trial y_p

$f(x) = 2e^{3x} = Ce^{3x}$

try $y_p = Ae^{3x}$

$y_p = Ae^{3x}$

$y_p' = 3Ae^{3x}$

$y_p'' = 9Ae^{3x}$

Want $y_p'' - y_p' - 2y_p = 2e^{3x}$ $\qquad (y'' - y' - 2y = 2e^{3x})$

$$9Ae^{3x} - 3Ae^{3x} - 2Ae^{3x} = 2e^{3x}$$

$$4Ae^{3x} = 2e^{3x}$$

$$4A = 2 \implies A = 1/2$$

$$\boxed{y_p = \frac{1}{2}e^{3x}}$$

Step 3 Combine, gen sol $y = \boxed{y_c} + \boxed{y_p}$

$$\boxed{y = \boxed{c_1 e^{-x} + c_2 e^{2x}} + \boxed{\frac{1}{2}e^{3x}}}$$

- $f(x)$ has x^k (incl. x^0, a constant) $\implies$ include $x^k, x^{k-1}, \dots, x, x^0 (=1)$ in trial y_p.

- $f(x)$ has e^{kx}, then include "Ae^{kx}" in y_p

- if $f(x)$ has $\cos(kx)$ or $\sin(kx)$ $\implies$ include BOTH $A\cos(kx)$ and $B\sin(kx)$ in y_p

$\underline{Ex}$: $y'' - y' - 2y = \cos x$

$\underline{\text{Step 1}}$ y_c : gen sol for $y'' - y' - 2y = 0$

same as last example

$$\underline{y_c = c_1 e^{-x} + c_2 e^{2x}}$$

$\underline{\text{Step 2}}$ $f(x) = \cos(1x)$, so trial

$$\underline{y_p = A\cos x + B\sin x}$$

$$y_p' = -A\sin x + B\cos x$$

$$y_p'' = -A\cos x - B\sin x$$

plug into $y_p'' - y_p' - 2y_p = \cos x$

$$(-A\cos x - B\sin x) - (-A\sin x + B\cos x) - 2(A\cos x + B\sin x)$$

$$(-3A - B)\cos x + (A - 3B)\sin x = \cos x$$

$$\left.\begin{aligned} -3A - B &= 1 \\ A - 3B &= 0 \end{aligned}\right\} \Rightarrow B = \frac{-1}{10},\ A = -\frac{3}{10}$$

so $\boxed{y_p = -\frac{3}{10}\cos x - \frac{1}{10}\sin x}$ is a solution (particular) for $y'' - y' - 2y = \cos x$

$\underline{\text{Step 3}}$ Gen solution for $y'' - y' - 2y = \cos x$ is $y = y_c + y_p = c_1 e^{-x} + \ldots$

Ex: $y'' - y' - 2y = \underbrace{x e^{3x}}_{f(x)}$

deg 1 polynomial $(1x+0)$ exponential e^{3x}

trial function $y_p = (A+Bx)e^{3x}$

Ex: $y'' - y' - 2y = x\sin(2x)$

trial function y_p

$$= (A+Bx)(C\cos(2x) + D\sin(2x))$$

$$AC\boxed{\cos(2x)} + BC\boxed{x\cos(2x)} + AD\boxed{\sin(2x)} + BD\boxed{x\sin(2x)}$$

really let's use

$$y_p = A\cos(2x) + Bx\cos(2x) + C\sin(2x) + Dx\sin(2x)$$